

# Smarandache Isotopy Theory Of Smarandache: Quasigroups And Loops <sup>\*†</sup>

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## Abstract

The concept of Smarandache isotopy is introduced and its study is explored for Smarandache: groupoids, quasigroups and loops just like the study of isotopy theory was carried out for groupoids, quasigroups and loops. The exploration includes: Smarandache; isotopy and isomorphy classes, Smarandache  $f, g$  principal isotopes and G-Smarandache loops.

## 1 Introduction

In 2002, W. B. Vasantha Kandasamy initiated the study of Smarandache loops in her book [12] where she introduced over 75 Smarandache concepts in loops. In her paper [13], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. For more on loops and their properties, readers should check [11], [1], [3], [4], [5] and [12]. In [[12], Page 102], the author introduced Smarandache isotopes of loops particularly Smarandache principal isotopes. She has also introduced the Smarandache concept in some other algebraic structures as [14, 15, 16, 17, 18, 19] account. The present author has contributed to the study of S-quasigroups and S-loops in [6], [7] and [8] while Muktibodh [10] did a study on the first.

In this study, the concept of Smarandache isotopy will be introduced and its study will be explored in Smarandache: groupoids, quasigroups and loops just like the study of isotopy theory was carried out for groupoids, quasigroups and loops as summarized in Bruck [1], Dene and Keedwell [4], Pflugfelder [11].

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## 2 Definitions and Notations

**Definition 2.1** Let  $L$  be a non-empty set. Define a binary operation  $(\cdot)$  on  $L$  : If  $x \cdot y \in L \forall x, y \in L$ ,  $(L, \cdot)$  is called a groupoid. If the system of equations  $a \cdot x = b$  and  $y \cdot a = b$  have unique solutions for  $x$  and  $y$  respectively, then  $(L, \cdot)$  is called a quasigroup. Furthermore, if there exists a unique element  $e \in L$  called the identity element such that  $\forall x \in L, x \cdot e = e \cdot x = x$ ,  $(L, \cdot)$  is called a loop.

If there exists at least a non-empty and non-trivial subset  $M$  of a groupoid(quasigroup or semigroup or loop)  $L$  such that  $(M, \cdot)$  is a non-trivial subsemigroup(subgroup or subgroup or subgroup) of  $(L, \cdot)$ , then  $L$  is called a Smarandache: groupoid( $S$ -groupoid)(quasigroup( $S$ -quasigroup) or semigroup( $S$ -semigroup) or loop( $S$ -loop)) with Smarandache: subsemigroup( $S$ -subsemigroup)(subgroup( $S$ -subgroup) or subgroup( $S$ -subgroup) or subgroup( $S$ -subgroup))  $M$ .

Let  $(G, \cdot)$  be a quasigroup(loop). The bijection  $L_x : G \rightarrow G$  defined as  $yL_x = x \cdot y \forall x, y \in G$  is called a left translation(multiplication) of  $G$  while the bijection  $R_x : G \rightarrow G$  defined as  $yR_x = y \cdot x \forall x, y \in G$  is called a right translation(multiplication) of  $G$ .

The set  $SYM(L, \cdot) = SYM(L)$  of all bijections in a groupoid  $(L, \cdot)$  forms a group called the permutation(symmetric) group of the groupoid  $(L, \cdot)$ .

**Definition 2.2** If  $(L, \cdot)$  and  $(G, \circ)$  are two distinct groupoids, then the triple  $(U, V, W) : (L, \cdot) \rightarrow (G, \circ)$  such that  $U, V, W : L \rightarrow G$  are bijections is called an isotopism if and only if

$$xU \circ yV = (x \cdot y)W \forall x, y \in L.$$

So we call  $L$  and  $G$  groupoid isotopes. If  $L = G$  and  $W = I$ (identity mapping) then  $(U, V, I)$  is called a principal isotopism, so we call  $G$  a principal isotope of  $L$ . But if in addition  $G$  is a quasigroup such that for some  $f, g \in G$ ,  $U = R_g$  and  $V = L_f$ , then  $(R_g, L_f, I) : (G, \cdot) \rightarrow (G, \circ)$  is called an  $f, g$ -principal isotopism while  $(G, \cdot)$  and  $(G, \circ)$  are called quasigroup isotopes.

If  $U = V = W$ , then  $U$  is called an isomorphism, hence we write  $(L, \cdot) \cong (G, \circ)$ . A loop  $(L, \cdot)$  is called a  $G$ -loop if and only if  $(L, \cdot) \cong (G, \circ)$  for all loop isotopes  $(G, \circ)$  of  $(L, \cdot)$ .

Now, if  $(L, \cdot)$  and  $(G, \circ)$  are  $S$ -groupoids with  $S$ -subsemigroups  $L'$  and  $G'$  respectively such that  $(G')A = L'$ , where  $A \in \{U, V, W\}$ , then the isotopism  $(U, V, W) : (L, \cdot) \rightarrow (G, \circ)$  is called a Smarandache isotopism( $S$ -isotopism). Consequently, if  $W = I$  the triple  $(U, V, I)$  is called a Smarandache principal isotopism. But if in addition  $G$  is a  $S$ -quasigroup with  $S$ -subgroup  $H'$  such that for some  $f, g \in H$ ,  $U = R_g$  and  $V = L_f$ , and  $(R_g, L_f, I) : (G, \cdot) \rightarrow (G, \circ)$  is an isotopism, then the triple is called a Smarandache  $f, g$ -principal isotopism while  $f$  and  $g$  are called Smarandache elements( $S$ -elements).

Thus, if  $U = V = W$ , then  $U$  is called a Smarandache isomorphism, hence we write  $(L, \cdot) \simeq (G, \circ)$ . An  $S$ -loop  $(L, \cdot)$  is called a  $G$ -Smarandache loop( $GS$ -loop) if and only if  $(L, \cdot) \simeq (G, \circ)$  for all loop isotopes(or particularly all  $S$ -loop isotopes)  $(G, \circ)$  of  $(L, \cdot)$ .

**Example 2.1** The systems  $(L, \cdot)$  and  $(L, *)$ ,  $L = \{0, 1, 2, 3, 4\}$  with the multiplication tables below are  $S$ -quasigroups with  $S$ -subgroups  $(L', \cdot)$  and  $(L'', *)$  respectively,  $L' = \{0, 1\}$  and  $L'' = \{1, 2\}$ .  $(L, \cdot)$  is taken from Example 2.2 of [10]. The triple  $(U, V, W)$  such that

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 & 3 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 0 & 4 & 3 \end{pmatrix}$$

are permutations on  $L$ , is an  $S$ -isotopism of  $(L, \cdot)$  onto  $(L, *)$ . Notice that  $A(L') = L''$  for all  $A \in \{U, V, W\}$  and  $U, V, W : L' \rightarrow L''$  are all bijections.

$\cdot$	0	1	2	3	4
0	0	1	3	4	2
1	1	0	2	3	4
2	3	4	1	2	0
3	4	2	0	1	3
4	2	3	4	0	1

$*$	0	1	2	3	4
0	1	0	4	2	3
1	3	1	2	0	4
2	4	2	1	3	0
3	0	4	3	1	2
4	2	3	0	4	1

**Example 2.2** According to Example 4.2.2 of [15], the system  $(\mathbb{Z}_6, \times_6)$  i.e the set  $L = \mathbb{Z}_6$  under multiplication modulo 6 is an  $S$ -semigroup with  $S$ -subgroups  $(L', \times_6)$  and  $(L'', \times_6)$ ,  $L' = \{2, 4\}$  and  $L'' = \{1, 5\}$ . This can be deduced from its multiplication table, below. The triple  $(U, V, W)$  such that

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 & 0 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 5 & 4 & 2 & 3 \end{pmatrix}$$

are permutations on  $L$ , is an  $S$ -isotopism of  $(\mathbb{Z}_6, \times_6)$  unto an  $S$ -semigroup  $(\mathbb{Z}_6, *)$  with  $S$ -subgroups  $(L''', *)$  and  $(L''', *)$ ,  $L''' = \{2, 5\}$  and  $L'''' = \{0, 3\}$  as shown in the second table below. Notice that  $A(L') = L'''$  and  $A(L'') = L''''$  for all  $A \in \{U, V, W\}$  and  $U, V, W : L' \rightarrow L'''$  and  $U, V, W : L'' \rightarrow L''''$  are all bijections.

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	1	1	4	4	1
2	5	1	5	2	1	2
3	3	1	5	0	4	2
4	1	1	1	1	1	1
5	2	1	2	5	1	5

**Remark 2.1** Taking careful look at Definition 2.2 and comparing it with [Definition 4.4.1, [12]], it will be observed that the author did not allow the component bijections  $U, V$  and  $W$  in  $(U, V, W)$  to act on the whole  $S$ -loop  $L$  but only on the  $S$ -subloop( $S$ -subgroup)  $L'$ . We feel this is necessary to adjust here so that the set  $L - L'$  is not out of the study. Apart from this, our adjustment here will allow the study of Smarandache isotopy to be

explorable. Therefore, the  $S$ -isotopism and  $S$ -isomorphism here are clearly special types of relations(isotopism and isomorphism) on the whole domain into the whole co-domain but those of Vasantha Kandasamy [12] only take care of the structure of the elements in the  $S$ -subloop and not the  $S$ -loop. Nevertheless, we do not fault her study for we think she defined them to apply them to some life problems as an applied algebraist.

### 3 Smarandache Isotopy and Isomorphy Classes

**Theorem 3.1** Let  $\mathfrak{G} = \left\{ (G_\omega, \circ_\omega) \right\}_{\omega \in \Omega}$  be a set of distinct  $S$ -groupoids with a corresponding set of  $S$ -subsemigroups  $\mathfrak{H} = \left\{ (H_\omega, \circ_\omega) \right\}_{\omega \in \Omega}$ . Define a relation  $\sim$  on  $\mathfrak{G}$  such that for all  $(G_{\omega_i}, \circ_{\omega_i}), (G_{\omega_j}, \circ_{\omega_j}) \in \mathfrak{G}$ , where  $\omega_i, \omega_j \in \Omega$ ,

$$(G_{\omega_i}, \circ_{\omega_i}) \sim (G_{\omega_j}, \circ_{\omega_j}) \iff (G_{\omega_i}, \circ_{\omega_i}) \text{ and } (G_{\omega_j}, \circ_{\omega_j}) \text{ are } S\text{-isotopic.}$$

Then  $\sim$  is an equivalence relation on  $\mathfrak{G}$ .

#### Proof

Let  $(G_{\omega_i}, \circ_{\omega_i}), (G_{\omega_j}, \circ_{\omega_j}), (G_{\omega_k}, \circ_{\omega_k}) \in \mathfrak{G}$ , where  $\omega_i, \omega_j, \omega_k \in \Omega$ .

**Reflexivity** If  $I : G_{\omega_i} \rightarrow G_{\omega_i}$  is the identity mapping, then

$$xI \circ_{\omega_i} yI = (x \circ_{\omega_i} y)I \quad \forall x, y \in G_{\omega_i} \implies \text{the triple } (I, I, I) : (G_{\omega_i}, \circ_{\omega_i}) \rightarrow (G_{\omega_i}, \circ_{\omega_i})$$

is an  $S$ -isotopism since  $(H_{\omega_i})I = H_{\omega_i} \quad \forall \omega_i \in \Omega$ . In fact, it can be simply deduced that every  $S$ -groupoid is  $S$ -isomorphic to itself.

**Symmetry** Let  $(G_{\omega_i}, \circ_{\omega_i}) \sim (G_{\omega_j}, \circ_{\omega_j})$ . Then there exist bijections

$$U, V, W : (G_{\omega_i}, \circ_{\omega_i}) \longrightarrow (G_{\omega_j}, \circ_{\omega_j}) \text{ such that } (H_{\omega_i})A = H_{\omega_j} \quad \forall A \in \{U, V, W\}$$

so that the triple

$$\alpha = (U, V, W) : (G_{\omega_i}, \circ_{\omega_i}) \longrightarrow (G_{\omega_j}, \circ_{\omega_j})$$

is an isotopism. Since each of  $U, V, W$  is bijective, then their inverses

$$U^{-1}, V^{-1}, W^{-1} : (G_{\omega_j}, \circ_{\omega_j}) \longrightarrow (G_{\omega_i}, \circ_{\omega_i})$$

are bijective. In fact,  $(H_{\omega_j})A^{-1} = H_{\omega_i} \quad \forall A \in \{U, V, W\}$  since  $A$  is bijective so that the triple

$$\alpha^{-1} = (U^{-1}, V^{-1}, W^{-1}) : (G_{\omega_j}, \circ_{\omega_j}) \longrightarrow (G_{\omega_i}, \circ_{\omega_i})$$

is an isotopism. Thus,  $(G_{\omega_j}, \circ_{\omega_j}) \sim (G_{\omega_i}, \circ_{\omega_i})$ .

**Transitivity** Let  $(G_{\omega_i}, \circ_{\omega_i}) \sim (G_{\omega_j}, \circ_{\omega_j})$  and  $(G_{\omega_j}, \circ_{\omega_j}) \sim (G_{\omega_k}, \circ_{\omega_k})$ . Then there exist bijections

$$U_1, V_1, W_1 : (G_{\omega_i}, \circ_{\omega_i}) \longrightarrow (G_{\omega_j}, \circ_{\omega_j}) \text{ and } U_2, V_2, W_2 : (G_{\omega_j}, \circ_{\omega_j}) \longrightarrow (G_{\omega_k}, \circ_{\omega_k})$$

$$\text{such that } (H_{\omega_i})A = H_{\omega_j} \ \forall A \in \{U_1, V_1, W_1\}$$

$$\text{and } (H_{\omega_j})B = H_{\omega_k} \ \forall B \in \{U_2, V_2, W_2\} \text{ so that the triples}$$

$$\alpha_1 = (U_1, V_1, W_1) : (G_{\omega_i}, \circ_{\omega_i}) \longrightarrow (G_{\omega_j}, \circ_{\omega_j}) \text{ and}$$

$$\alpha_2 = (U_2, V_2, W_2) : (G_{\omega_j}, \circ_{\omega_j}) \longrightarrow (G_{\omega_k}, \circ_{\omega_k})$$

are isotopisms. Since each of  $U_i, V_i, W_i, i = 1, 2$ , is bijective, then

$$U_3 = U_1U_2, V_3 = V_1V_2, W_3 = W_1W_2 : (G_{\omega_i}, \circ_{\omega_i}) \longrightarrow (G_{\omega_k}, \circ_{\omega_k})$$

are bijections such that  $(H_{\omega_i})A_3 = (H_{\omega_i})A_1A_2 = (H_{\omega_j})A_2 = H_{\omega_k}$  so that the triple

$$\alpha_3 = \alpha_1\alpha_2 = (U_3, V_3, W_3) : (G_{\omega_i}, \circ_{\omega_i}) \longrightarrow (G_{\omega_k}, \circ_{\omega_k})$$

is an isotopism. Thus,  $(G_{\omega_i}, \circ_{\omega_i}) \sim (G_{\omega_k}, \circ_{\omega_k})$ .

**Remark 3.1** As a follow up to Theorem 3.1, the elements of the set  $\mathfrak{S}/\sim$  will be referred to as Smarandache isotopy classes(*S-isotopy classes*). Similarly, if  $\sim$  meant "S-isomorphism" in Theorem 3.1, then the elements of  $\mathfrak{S}/\sim$  will be referred to as Smarandache isomorphism classes(*S-isomorphism classes*). Just like isotopy has an advantage over isomorphism in the classification of loops, so also S-isotopy will have advantage over S-isomorphism in the classification of S-loops.

**Corollary 3.1** Let  $\mathcal{L}_n$ ,  $\mathcal{SL}_n$  and  $\mathcal{NSL}_n$  be the sets of; all finite loops of order  $n$ ; all finite S-loops of order  $n$  and all finite non S-loops of order  $n$  respectively.

1. If  $\mathcal{A}_i^n$  and  $\mathcal{B}_i^n$  represent the isomorphism class of  $\mathcal{L}_n$  and the S-isomorphism class of  $\mathcal{SL}_n$  respectively, then

$$(a) \ |\mathcal{SL}_n| + |\mathcal{NSL}_n| = |\mathcal{L}_n|;$$

$$(i) \ |\mathcal{SL}_5| + |\mathcal{NSL}_5| = 56,$$

$$(ii) \ |\mathcal{SL}_6| + |\mathcal{NSL}_6| = 9,408 \text{ and}$$

$$(iii) \ |\mathcal{SL}_7| + |\mathcal{NSL}_7| = 16,942,080.$$

$$(b) \ |\mathcal{NSL}_n| = \sum_{i=1} |\mathcal{A}_i^n| - \sum_{i=1} |\mathcal{B}_i^n|;$$

$$(i) \ |\mathcal{NSL}_5| = \sum_{i=1}^6 |\mathcal{A}_i^5| - \sum_{i=1} |\mathcal{B}_i^5|,$$

$$(ii) \ |\mathcal{NSL}_6| = \sum_{i=1}^{109} |\mathcal{A}_i^6| - \sum_{i=1} |\mathcal{B}_i^6| \text{ and}$$

$$(iii) \ |\mathcal{NSL}_7| = \sum_{i=1}^{23,746} |\mathcal{A}_i^7| - \sum_{i=1} |\mathcal{B}_i^7|.$$

2. If  $\mathfrak{A}_i^n$  and  $\mathfrak{B}_i^n$  represent the isotopy class of  $\mathcal{L}_n$  and the  $S$ -isotopy class of  $\mathcal{SL}_n$  respectively, then

$$|\mathcal{NSL}_n| = \sum_{i=1} |\mathfrak{A}_i^n| - \sum_{i=1} |\mathfrak{B}_i^n|;$$

- (i)  $|\mathcal{NSL}_5| = \sum_{i=1}^2 |\mathfrak{A}_i^5| - \sum_{i=1} |\mathfrak{B}_i^5|,$
- (ii)  $|\mathcal{NSL}_6| = \sum_{i=1}^{22} |\mathfrak{A}_i^6| - \sum_{i=1} |\mathfrak{B}_i^6|$  and
- (iii)  $|\mathcal{NSL}_7| = \sum_{i=1}^{564} |\mathfrak{A}_i^7| - \sum_{i=1} |\mathfrak{B}_i^7|.$

**Proof**

An  $S$ -loop is an  $S$ -groupoid. Thus by Theorem 3.1, we have  $S$ -isomorphism classes and  $S$ -isotopy classes. Recall that  $|\mathcal{L}_n| = |\mathcal{SL}_n| + |\mathcal{NSL}_n| - |\mathcal{SL}_n \cap \mathcal{NSL}_n|$  but  $\mathcal{SL}_n \cap \mathcal{NSL}_n = \emptyset$  so  $|\mathcal{L}_n| = |\mathcal{SL}_n| + |\mathcal{NSL}_n|$ . As stated and shown in [11], [5], [2] and [9], the facts in Table 1 are true where  $n$  is the order of a finite loop. Hence the claims follow.

$n$	5	6	7
$ \mathcal{L}_n $	56	9, 408	16, 942, 080
$\{\mathcal{A}_i^n\}_{i=1}^k$	$k = 6$	$k = 109$	$k = 23, 746$
$\{\mathfrak{A}_i^n\}_{i=1}^m$	$m = 2$	$m = 22$	$m = 564$

Table 1: Enumeration of Isomorphism and Isotopy classes of finite loops of small order

**Question 3.1** How many  $S$ -loops are in the family  $\mathcal{L}_n$ ? That is, what is  $|\mathcal{SL}_n|$  or  $|\mathcal{NSL}_n|$ .

**Theorem 3.2** Let  $(G, \cdot)$  be a finite  $S$ -groupoid of order  $n$  with a finite  $S$ -subsemigroup  $(H, \cdot)$  of order  $m$ . Also, let

$$ISOT(G, \cdot), SISOT(G, \cdot) \text{ and } NSISOT(G, \cdot)$$

be the sets of all isotopisms,  $S$ -isotopisms and non  $S$ -isotopisms of  $(G, \cdot)$ . Then,

$$ISOT(G, \cdot) \text{ is a group and } SISOT(G, \cdot) \leq ISOT(G, \cdot).$$

Furthermore:

- 1.  $|ISOT(G, \cdot)| = (n!)^3;$
- 2.  $|SISOT(G, \cdot)| = (m!)^3;$
- 3.  $|NSISOT(G, \cdot)| = (n!)^3 - (m!)^3.$

**Proof**

1. This has been shown to be true in [Theorem 4.1.1, [4]].
2. An S-isotopism is an isotopism. So,  $\mathcal{SISOT}(G, \cdot) \subset \mathcal{ISOT}(G, \cdot)$ . Thus, we need to just verify the axioms of a group to show that  $\mathcal{SISOT}(G, \cdot) \leq \mathcal{ISOT}(G, \cdot)$ . These can be done using the proofs of reflexivity, symmetry and transitivity in Theorem 3.1 as guides. For all triples

$$\alpha \in \mathcal{SISOT}(G, \cdot) \text{ such that } \alpha = (U, V, W) : (G, \cdot) \longrightarrow (G, \circ),$$

where  $(G, \cdot)$  and  $(G, \circ)$  are S-groupoids with S-subgroups  $(H, \cdot)$  and  $(K, \circ)$  respectively, we can set

$$U' := U|_H, V' := V|_H \text{ and } W' := W|_H \text{ since } A(H) = K \ \forall A \in \{U, V, W\},$$

so that  $\mathcal{SISOT}(H, \cdot) = \{(U', V', W')\}$ . This is possible because of the following arguments.

Let

$$X = \left\{ f' := f|_H \mid f : G \longrightarrow G, f : H \longrightarrow K \text{ is bijective and } f(H) = K \right\}.$$

Let

$$SYM(H, K) = \{\text{bijections from } H \text{ unto } K\}.$$

By definition, it is easy to see that  $X \subseteq SYM(H, K)$ . Now, for all  $U \in SYM(H, K)$ , define  $U : H^c \longrightarrow K^c$  so that  $U : G \longrightarrow G$  is a bijection since  $|H| = |K|$  implies  $|H^c| = |K^c|$ . Thus,  $SYM(H, K) \subseteq X$  so that  $SYM(H, K) = X$ .

Given that  $|H| = m$ , then it follows from (1) that

$$|\mathcal{ISOT}(H, \cdot)| = (m!)^3 \text{ so that } |\mathcal{SISOT}(G, \cdot)| = (m!)^3 \text{ since } SYM(H, K) = X.$$

3.

$$\mathcal{NSISOT}(G, \cdot) = (\mathcal{SISOT}(G, \cdot))^c.$$

So, the identity isotopism

$$(I, I, I) \notin \mathcal{NSISOT}(G, \cdot), \text{ hence } \mathcal{NSISOT}(G, \cdot) \not\leq \mathcal{ISOT}(G, \cdot).$$

Furthermore,

$$|\mathcal{NSISOT}(G, \cdot)| = (n!)^3 - (m!)^3.$$

**Corollary 3.2** *Let  $(G, \cdot)$  be a finite S-groupoid of order  $n$  with an S-subsemigroup  $(H, \cdot)$ . If  $\mathcal{ISOT}(G, \cdot)$  is the group of all isotopisms of  $(G, \cdot)$  and  $S_n$  is the symmetric group of degree  $n$ , then*

$$\mathcal{ISOT}(G, \cdot) \simeq S_n \times S_n \times S_n.$$

**Proof**

As concluded in [Corollary 1, [4]],  $ISOT(G, \cdot) \cong S_n \times S_n \times S_n$ . Let  $PISOT(G, \cdot)$  be the set of all principal isotopisms on  $(G, \cdot)$ .  $PISOT(G, \cdot)$  is an S-subgroup in  $ISOT(G, \cdot)$  while  $S_n \times S_n \times \{I\}$  is an S-subgroup in  $S_n \times S_n \times S_n$ . If

$$\Upsilon : ISOT(G, \cdot) \longrightarrow S_n \times S_n \times S_n \text{ is defined as}$$

$$\Upsilon((A, B, I)) = \langle A, B, I \rangle \quad \forall (A, B, I) \in ISOT(G, \cdot),$$

then

$$\Upsilon(PISOT(G, \cdot)) = S_n \times S_n \times \{I\}. \quad \therefore ISOT(G, \cdot) \simeq S_n \times S_n \times S_n.$$

## 4 Smarandache $f, g$ -Isotopes of Smarandache Loops

**Theorem 4.1** *Let  $(G, \cdot)$  and  $(H, *)$  be S-groupoids. If  $(G, \cdot)$  and  $(H, *)$  are S-isotopic, then  $(H, *)$  is S-isomorphic to some Smarandache principal isotope  $(G, \circ)$  of  $(G, \cdot)$ .*

**Proof**

Since  $(G, \cdot)$  and  $(H, *)$  are S-isotopic S-groupoids with S-subsemigroups  $(G_1, \cdot)$  and  $(H_1, *)$ , then there exist bijections  $U, V, W : (G, \cdot) \rightarrow (H, *)$  such that the triple  $\alpha = (U, V, W) : (G, \cdot) \rightarrow (H, *)$  is an isotopism and  $(G_1)A = H_1 \quad \forall A \in \{U, V, W\}$ . To prove the claim of this theorem, it suffices to produce a closed binary operation  $'\circ'$  on  $G$ , bijections  $X, Y : G \rightarrow G$ , and bijection  $Z : G \rightarrow H$  so that

- the triple  $\beta = (X, Y, I) : (G, \cdot) \rightarrow (G, \circ)$  is a Smarandache principal isotopism and
- $Z : (G, \circ) \rightarrow (H, *)$  is an S-isomorphism or the triple  $\gamma = (Z, Z, Z) : (G, \circ) \rightarrow (H, *)$  is an S-isotopism.

Thus, we need  $(G, \circ)$  so that the commutative diagram below is true:

$$\begin{array}{ccc} (G, \cdot) & \xrightarrow[\text{isotopism}]{\alpha} & (H, *) \\ & \searrow \beta & \uparrow \gamma \text{ isomorphism} \\ & & (G, \circ) \end{array}$$

principal isotopism

because following the proof of transitivity in Theorem 3.1,  $\alpha = \beta\gamma$  which implies  $(U, V, W) = (XZ, YZ, Z)$  and so we can make the choices;  $Z = W$ ,  $Y = VW^{-1}$ , and  $X = UW^{-1}$  and consequently,

$$x \cdot y = xUW^{-1} \circ VW^{-1} \iff x \circ y = xWU^{-1} \cdot yWV^{-1} \quad \forall x, y \in G.$$

Hence,  $(G, \circ)$  is a groupoid principal isotope of  $(G, \cdot)$  and  $(H, *)$  is an isomorph of  $(G, \circ)$ . It remains to show that these two relationships are Smarandache.

Note that  $((H_1)Z^{-1}, \circ) = (G_1, \circ)$  is a non-trivial subsemigroup in  $(G, \circ)$ . Thus,  $(G, \circ)$  is an S-groupoid. So  $(G, \circ) \simeq (H, *)$ .  $(G, \cdot)$  and  $(G, \circ)$  are Smarandache principal isotopes because  $(G_1)UW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1$  and  $(G_1)VW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1$ .



**Corollary 4.1** *Let  $(G, \cdot)$  be an S-groupoid with an arbitrary groupoid isotope  $(H, *)$ . Any such groupoid  $(H, *)$  is an S-groupoid if and only if all the principal isotopes of  $(G, \cdot)$  are S-groupoids.*

**Proof**

By classical result in principal isotopy [[11], III.1.4 Theorem], if  $(G, \cdot)$  and  $(H, *)$  are isotopic groupoids, then  $(H, *)$  is isomorphic to some principal isotope  $(G, \circ)$  of  $(G, \cdot)$ . Assuming  $(H, *)$  is an S-groupoid then since  $(H, *) \cong (G, \circ)$ ,  $(G, \circ)$  is an S-groupoid. Conversely, let us assume all the principal isotopes of  $(G, \cdot)$  are S-groupoids. Since  $(H, *) \cong (G, \circ)$ , then  $(H, *)$  is an S-groupoid.

**Theorem 4.2** *Let  $(G, \cdot)$  be an S-quasigroup. If  $(H, *)$  is an S-loop which is S-isotopic to  $(G, \cdot)$ , then there exist S-elements  $f$  and  $g$  so that  $(H, *)$  is S-isomorphic to a Smarandache  $f, g$  principal isotope  $(G, \circ)$  of  $(G, \cdot)$ .*

**Proof**

An S-quasigroup and an S-loop are S-groupoids. So by Theorem 4.1,  $(H, *)$  is S-isomorphic to a Smarandache principal isotope  $(G, \circ)$  of  $(G, \cdot)$ . Let  $\alpha = (U, V, I)$  be the Smarandache principal isotopism of  $(G, \cdot)$  onto  $(G, \circ)$ . Since  $(H, *)$  is a S-loop and  $(G, \circ) \succsim (H, *)$  implies that  $(G, \circ) \cong (H, *)$ , then  $(G, \circ)$  is necessarily an S-loop and consequently,  $(G, \circ)$  has a two-sided identity element say  $e$  and an S-subgroup  $(G_2, \circ)$ . Let  $\alpha = (U, V, I)$  be the Smarandache principal isotopism of  $(G, \cdot)$  onto  $(G, \circ)$ . Then,

$$xU \circ yV = x \cdot y \quad \forall x, y \in G \iff x \circ y = xU^{-1} \cdot yV^{-1} \quad \forall x, y \in G.$$

So,

$$y = e \circ y = eU^{-1} \cdot yV^{-1} = yV^{-1}L_{eU^{-1}} \quad \forall y \in G \text{ and } x = x \circ e = xU^{-1} \cdot eV^{-1} = xU^{-1}R_{eV^{-1}} \quad \forall x \in G.$$

Assign  $f = eU^{-1}, g = eV^{-1} \in G_2$ . This assignments are well defined and hence  $V = L_f$  and  $U = R_g$ . So that  $\alpha = (R_g, L_f, I)$  is a Smarandache  $f, g$  principal isotopism of  $(G, \circ)$  onto  $(G, \cdot)$ . This completes the proof.

**Corollary 4.2** *Let  $(G, \cdot)$  be an S-quasigroup(S-loop) with an arbitrary groupoid isotope  $(H, *)$ . Any such groupoid  $(H, *)$  is an S-quasigroup(S-loop) if and only if all the principal isotopes of  $(G, \cdot)$  are S-quasigroups(S-loops).*

**Proof**

This follows immediately from Corollary 4.1 since an S-quasigroup and an S-loop are S-groupoids.

**Corollary 4.3** *If  $(G, \cdot)$  and  $(H, *)$  are S-loops which are S-isotopic, then there exist S-elements  $f$  and  $g$  so that  $(H, *)$  is S-isomorphic to a Smarandache  $f, g$  principal isotope  $(G, \circ)$  of  $(G, \cdot)$ .*

**Proof**

An S-loop is an S-quasigroup. So the claim follows from Theorem 4.2.

## 5 G-Smarandache Loops

**Lemma 5.1** *Let  $(G, \cdot)$  and  $(H, *)$  be S-isotopic S-loops. If  $(G, \cdot)$  is a group, then  $(G, \cdot)$  and  $(H, *)$  are S-isomorphic groups.*

**Proof**

By Corollary 4.3, there exist S-elements  $f$  and  $g$  in  $(G, \cdot)$  so that  $(H, *) \succsim (G, \circ)$  such that  $(G, \circ)$  is a Smarandache  $f, g$  principal isotope of  $(G, \cdot)$ .

Let us set the mapping  $\psi := R_{f \cdot g} = R_{fg} : G \rightarrow G$ . This mapping is bijective. Now, let us consider when  $\psi := R_{fg} : (G, \cdot) \rightarrow (G, \circ)$ . Since  $(G, \cdot)$  is associative and  $x \circ y = xR_g^{-1} \cdot yL_f^{-1} \forall x, y \in G$ , the following arguments are true.

$x\psi \circ y\psi = x\psi R_g^{-1} \cdot y\psi L_f^{-1} = xR_{fg}R_g^{-1} \cdot yR_{fg}L_f^{-1} = x \cdot fg \cdot g^{-1} \cdot f^{-1} \cdot y \cdot fg = x \cdot y \cdot fg = (x \cdot y)R_{fg} = (x \cdot y)\psi \forall x, y \in G$ . So,  $(G, \cdot) \cong (G, \circ)$ . Thus,  $(G, \circ)$  is a group. If  $(G_1, \cdot)$  and  $(G_1, \circ)$  are the S-subgroups in  $(G, \cdot)$  and  $(G, \circ)$ , then  $((G_1, \cdot))R_{fg} = (G_1, \circ)$ . Hence,  $(G, \cdot) \succsim (G, \circ)$ .

$\therefore (G, \cdot) \succsim (H, *)$  and  $(H, *)$  is a group.

**Corollary 5.1** *Every group which is an S-loop is a GS-loop.*

**Proof**

This follows immediately from Lemma 5.1 and the fact that a group is a G-loop.

**Corollary 5.2** *An S-loop is S-isomorphic to all its S-loop S-isotopes if and only if it is S-isomorphic to all its Smarandache  $f, g$  principal isotopes.*

**Proof**

Let  $(G, \cdot)$  be an S-loop with arbitrary S-isotope  $(H, *)$ . Let us assume that  $(G, \cdot) \succsim (H, *)$ . From Corollary 4.3, for any arbitrary S-isotope  $(H, *)$  of  $(G, \cdot)$ , there exists a Smarandache  $f, g$  principal isotope  $(G, \circ)$  of  $(G, \cdot)$  such that  $(H, *) \succsim (G, \circ)$ . So,  $(G, \cdot) \succsim (G, \circ)$ .

Conversely, let  $(G, \cdot) \succsim (G, \circ)$ , using the fact in Corollary 4.3 again, for any arbitrary S-isotope  $(H, *)$  of  $(G, \cdot)$ , there exists a Smarandache  $f, g$  principal isotope  $(G, \circ)$  of  $(G, \cdot)$  such that  $(G, \circ) \succsim (H, *)$ . Therefore,  $(G, \cdot) \succsim (H, *)$ .

**Corollary 5.3** *A S-loop is a GS-loop if and only if it is S-isomorphic to all its Smarandache  $f, g$  principal isotopes.*

**Proof**

This follows by the definition of a GS-loop and Corollary 5.2.

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